14^{th} National Conference on Machines and Mechanisms (NaCoMM - 09),
NIT, Durgapur, India. December 17-18, 2009 **NaCoMM - 09-Paper ID: RAM08**

Homogeneous Matrix Approach based on Joint Motion for Forward Kinematic Analysis of Serial Mechanisms

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Abstract

The paper deals with the concept of the alternative derivation of homogeneous position transformation matrix $[T]$ from the point of view of vector mechanics of translation and rotation of base frame to body frame. A joint motion property vector $\{D\}$ is defined. The $\{D\}$-vector and its time-derivatives are represented in matrix form. $[D]$-matrix and its time derivatives are used in deriving homogenous velocity transformation matrix and homogeneous acceleration transformation matrix. Jacobian and its time-derivatives are also derived. The method is found to be consistent with loop-closure-based approach and the results are consistent with the approach based on relative velocity and relative acceleration. The method has also been extended to jerk calculations. This investigation leads to a computationally efficient alternative method for forward kinematic analysis of serial mechanisms.
Keywords: Joint motion property vector / matrix.

1 Introduction

The forward kinematics is an well established topic. We get several discussion on this topic on robotics and mechanisms. Conventional approach is to establish homogeneous position transformation matrix from the point of view of direction cosines and then to use vector analysis that leads to the matrix formulation. Selig [5] in his book discussed various ways of analysis for serial robots using *LieAlgebra*. Shabana [6] in his book discussed in details the fundamental aspects of kinematics of rigid multi-body system. Jazar [2]also in his book discussed at length various approaches for the analysis of serial robots. Huston [1] elaborated the approach of partial velocity based on generalized coordinates to deal with the kinematics of rigid body. Saha [4] in his book defined the concept of Decoupled Natural Orthogonal Complement matrix (DENOC) and developed an approach based on it.

Author found some matrix formulations in text books by and Shahinpoor [7] and Uicker [8]. The derivation were based on concept of well known loop closure equation in old text books on mechanisms or robotics. While using vector mechanics without the concept of loop closure equation, author verified results mentioned in earlier text as well as found some interesting conclusions. The results lead to a well balanced approach for entire forward kinematics of serial mechanism in general. It gives insight into the mechanics and is also computationally efficient.

Basic Idea:

The motion of a rigid body in general is complex. The complex motion of a rigid body can be explained by Chasles' theorem. The statement of Chasles' theorem as expressed by Norton [3] is presented below.

Chasles' Theorem: Any displacement of a rigid body is equivalent to the sum of a translation of any point on that body and a rotation of the body about an axis through that point.

The theorem is valid for velocity also. The axis referred in the theorem will be known as *motion axis* in this paper. With respect this axis, the motion of a rigid body is specified.

On the other hand, the position of a point in a rigid body is specified by a constant position vector (which does not change with respect time) in a reference frame fixed with the body.

So there are two reference frames. One reference frame is fixed with the body in motion and called body frame. Another reference frame, with respect to which the orientation and location of the motion axis are fixed and specified. This reference frame is called motion frame. So it may be concluded that there exist another reference frame with respect to which other two reference frames mentioned above will be specified. This reference frame is seldom inertial reference frame at rest and is called base frame. *In this paper the Base frame is a inertial frame and is considered at rest.*

Motion is given / specified along the motion axis in the form of translation along the axis or rotation about the axis or a combination of both. The motions are quantitatively specified with respect to motion axis that is fixed in motion frame. In every serial mechanism, there exists a system of propagation of motion and frames are numbered accordingly.

For a n-link serial manipulator robot, the motion is transferred from base to the end-effector. The base body is numbered as 1 and base frame which is fixed to the base body is also numbered as 1-st frame. The end effector is numbered as n-th link. The frame fixed to the end-effector is numbered as n-th frame. This frame is body frame and is used to specify the geometry of end-effector. The (n-1)-th frame which is fixed with (n-1)-th body is body frame to specify the geometry of (n-1)-th body. The (n-1)-th frame is also motion frame for end-effector for the description of motion. In case of serial spur gear drive, at the center of a spur gear we have two frames - one fixed to the gear as body frame and another frame as motion frame.

Notation used:

- Let ${}^{i}\{Q\}_j$ denotes a constant vector (i.e. does not change with time) j-th frame and measured in i-th frame. In i-th frame the vector may change with time. If i-th frame is the same as j-th frame, the notation becomes ${}^{j}\{Q\}_j$. In this case superscript j is omitted and is simply written as $\{Q\}_j$.
- Base frame is denoted by 1-st frame and corresponding link/body is also denoted by link 1 or body 1.
- Homogeneous vector is denoted by subscript H.
- $[0]$ is 3×3 null square matrix and $\lfloor 0 \rfloor$ is 1×3 null row vector. $\{0\}$ is 3×1 null column vector. $[I]$ is square identity matrix.
- $\{u\}$, $\{v\}$ and $\{w\}$ indicate unit vectors in x-direction, y-direction and z-direction of right-hand orthogonal x-y-z frame.

Definition of cross-product matrix and some operators used in this paper:

Cross-product matrix:

$$\vec{a}\times = [\tilde{a}] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Notations as defined by the author for presentation of matrix form to vector form and vice-versa:

$$m4v3\left(\begin{bmatrix} 0 & -c & b & d \\ c & 0 & -a & e \\ -b & a & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

$$m4v6\left(\begin{bmatrix} 0 & -c & b & d \\ c & 0 & -a & e \\ -b & a & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{Bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{Bmatrix}$$

$$v6m4\left(\begin{Bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{Bmatrix}\right) = \begin{bmatrix} 0 & -c & b & d \\ c & 0 & -a & e \\ -b & a & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notation defined by the author for the presentation of 3-D homogeneous vector to 3-D vector (normal form).

$$v4v3\left(\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix}\right) = \begin{Bmatrix} a \\ b \\ c \end{Bmatrix}$$

Here,
$m4$ = 4×4 square matrix; $v6$ = 6×1 vector; $v4$ = 4×1 homogeneous vector; $v3$ = 3×1 vector.

2 Derivation of Homogeneous Position Transformation Matrix:

If a reference frame which initially coincides with the base reference frame is translated and rotated to coincide with body frame, the three unit vectors those represent the three orthogonal coordinate directions of the body frame with respect to base frame and a position vector that represents the origin of the body frame with respect to base frame can be computed from the concept of translation and rotation of vector.

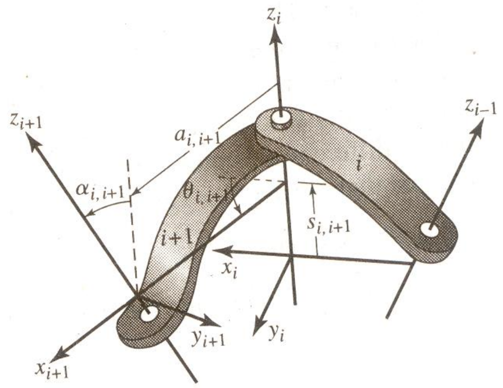

Figure 1: Definition of Denavit-Hartenberg Parameters (Uicker [8].)

In general we may consider the two frames - i-th frame and (i+1)-th frame of a multi-link serial robot. The frames are connected by (i+1)-th link. The geometry of the system is specified by Denavit-Hartenberg approach. According to DH approach, two frames are related and specified by two link parameters $a_{i,i+1}$ (link length) and $\alpha_{i,i+1}$ (link twist) and two joint variables $s_{i,i+1}$ (joint distance) and $\theta_{i,i+1}$ (joint angle). The (i+1)-th frame is fixed to link (i+1) and is the body frame for the description of the geometry of (i+1)-th link. The i-th frame is not fixed to the link i but not to the link i+1. The i-th frame is the motion frame for description of motion of link i. It requires mentioning that link parameters $a_{i,i+1}$ and $\alpha_{i,i+1}$ as well as joint variables $s_{i,i+1}$ and $\theta_{i,i+1}$ are also referred in short as a_i, α_i, s_i and θ_i respectively.

Let $\{u\}_i$, $\{v\}_i$, $\{w\}_i$ and $\{r\}_i$ represent the three unit vectors of three co-ordinate axes and the position vector of the origin of i-th frame as measured in i-th frame and $\{u\}_{i+1}$, $\{v\}_{i+1}$, $\{w\}_{i+1}$ and $\{r\}_{i+1}$are the three unit vectors of three co-ordinate axes and the position vector of the origin for (i+1)-th frame as measured in (i+1)-th frame. ${}^{i}\{r\}_{i+1}$ is the origin of (i+1)-th frame as measured in i-th frame and $\{r\}_i$ is the origin of i-th frame as measured in i-th frame. ${}^{i}\{u\}_{i+1}$, ${}^{i}\{v\}_{i+1}$ and ${}^{i}\{w\}_{i+1}$ represent $\{u\}_{i+1}$ unit vector, $\{v\}_{i+1}$ unit vector and $\{w\}_{i+1}$ unit vector of (i+1)-th frame as measured in i-th frame and ${}^{i}\{r\}_{i+1}$ represents position vector of the origin of (i+1)-th frame as measured in i-th frame. This completes the information of (i+1)-th frame with respect to i-th frame.

Basic idea:

1. It is assumed that initially the (i+1)-th frame is coincident

totally with i-th frame.

2. Translation of (i+1)-th frame from i-th frame by $s_{i,i+1}$ along $\{w\}_i$ unit vector and rotation of frame about the same by $\theta_{i,i+1}$ and translation of frame by $a_{i,i+1}$ along the new orientation of x-axis of (i+1)-th frame. This process gives the position vector of the origin of (i+1)-th frame as measured in i-th frame,i.e. ${}^i\{r\}_{i+1}$ and the orientation of x-axis of (i+1)-th frame i.e. ${}^i\{u\}_{i+1}$ unit vector.

3. Rotation of (i+1)-th frame by angle $\alpha_{i,i+1}$ about ${}^i\{u\}_{i+1}$ unit vector (i.e. its own x-axis of (i+1)-th frame) gives orientations of ${}^i\{w\}_{i+1}$ (i.e. z-axis of (i+1)-th frame) and ${}^i\{v\}_{i+1}$ (i.e. y-axis of (i+1)-th frame).

The following mathematical steps in vector mechanics can generate ${}^i[T]_{i+1}$.

1. If $a_{i,i+1} \times \{u\}_i + 0 \times \{v\}_i + s_{i,i+1} \times \{w\}_i$ vector in i-th frame be rotated about $\{w\}_i$ unit vector by an angle $\theta_{i,i+1}$ and add to $\{r\}_i$, the relative position vector ${}^i\{r\}_{i+1}$ of origin of (i+1)-th frame with respect to origin of i-th frame is obtained.

2. If a vector $1 \times \{u\}_i + 0 \times \{v\}_i + 0\{w\}_i$ in i-th frame be rotated about $\{w\}_i$ unit vector by an angle $\theta_{i,i+1}$, ${}^i\{u\}_{i+1}$ unit vector of (i+1)-th frame with respect to i-th frame is obtained.

3. If a vector $0 \times \{u\}_i + 0 \times \{v\}_i + 1\{w\}_i$ in i-th frame be rotated about ${}^i\{u\}_{i+1}$ unit vector by an angle $\alpha_{i,i+1}$, ${}^i\{w\}_{i+1}$ unit vector of (i+1)-th frame with respect to i-th frame is obtained.

4. If vector $0 \times^i \{u\}_{i+1} + 0 \times^i \{v\}_{i+1} + 1 \times^i \{w\}_{i+1}$ in i-frame be rotated about ${}^i\{u\}_{i+1}$ by an angle $-\frac{\pi}{2}$ radian, we get ${}^i\{v\}_{i+1}$ unit vector of (i+1)-th frame with respect to i-th frame is obtained.

The rotation formula as suggested by Huston [1] may be used. Details of derivations are avoided.

$$
\begin{aligned}
{}^i\{r\}_{i+1} &= \begin{bmatrix} \{u\}_i & \{v\}_i & \{w\}_i & \{r\}_i \end{bmatrix} {}^i\{T_4\}_{i+1} \\
{}^i\{T_4\}_{i+1} &= \begin{bmatrix} a_{i,i+1}\cos\theta_{i,i+1} & a_{i,i+1}\sin\theta_{i,i+1} & s_{i,i+1} & 1 \end{bmatrix}^T \\
{}^i\{u\}_{i+1} &= \begin{bmatrix} \{u\}_i & \{v\}_i & \{w\}_i & \{r\}_i \end{bmatrix} {}^i\{T_1\}_{i+1} \\
{}^i\{T_1\}_{i+1} &= \begin{bmatrix} \cos\theta_{i,i+1} & \sin\theta_{i,i+1} & 0 & 0 \end{bmatrix}^T \\
{}^i\{w\}_{i+1} &= \begin{bmatrix} \{u\}_i & \{v\}_i & \{w\}_i & \{r\}_i \end{bmatrix} {}^i\{T_3\}_{i+1} \\
{}^i\{T_3\}_{i+1} &= \begin{bmatrix} -\cos\alpha_{i,i+1}\sin\theta_{i,i+1} & \cos\alpha_{i,i+1}\sin\theta_{i,i+1} & \sin\alpha_{i,i+1} & 0 \end{bmatrix}^T \\
{}^i\{v\}_{i+1} &= \begin{bmatrix} \{u\}_i & \{v\}_i & \{w\}_i & \{r\}_i \end{bmatrix} {}^i\{T_2\}_{i+1} \\
{}^i\{T_2\}_{i+1} &= \begin{bmatrix} a_{i,i+1}\cos\theta_{i,i+1} & -a_{i,i+1}\sin\theta_{i,i+1} & \cos\alpha_{i,i+1} & 1 \end{bmatrix}^T
\end{aligned}
$$

${}^i\{T_1\}_{i+1}$,${}^i\{T_2\}_{i+1}$, ${}^i\{T_3\}_{i+1}$ and ${}^i\{T_4\}_{i+1}$ are the first, second, third and forth column of Homogeneous Position transformation matrix ${}^i[T]_{i+1}$ respectively.

Thus,the homogeneous position transformation matrix from (i+1)-th frame to i-th frame is derived and presented below.

$$
{}^i[T]_{i+1} = \begin{bmatrix} {}^i\{T_1\}_{i+1} & {}^i\{T_2\}_{i+1} & {}^i\{T_3\}_{i+1} & {}^i\{T_4\}_{i+1} \end{bmatrix}
$$

we may write

$$
\begin{aligned}
&\begin{bmatrix} {}^i\{u\}_{i+1} & {}^i\{v\}_{i+1} & {}^i\{w\}_{i+1} & {}^i\{r\}_{i+1} \end{bmatrix} \\
&= \begin{bmatrix} \{u\}_i & \{v\}_i & \{w\}_i & \{r\}_i \end{bmatrix} {}^i[T]_{i+1}
\end{aligned}
$$

The above matrix presentation may be modified to suit homogeneous presentation of vectors,

$$
\begin{aligned}
&\begin{bmatrix} {}^i\{u\}_{i+1} & {}^i\{v\}_{i+1} & {}^i\{w\}_{i+1} & {}^i\{r\}_{i+1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \{u\}_i & \{v\}_i & \{w\}_i & \{r\}_i \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^i[T]_{i+1}
\end{aligned}
$$

The idea may be extended and generalized as,

$$
\begin{aligned}
&\begin{bmatrix} {}^i\{u\}_j & {}^i\{v\}_j & {}^i\{w\}_j & {}^i\{r\}_j \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \{u\}_i & \{v\}_i & \{w\}_i & \{r\}_i \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^i[T]_j
\end{aligned}
$$

where,

$$
{}^i[T]_j = \prod_{k=i}^{j-1} \left[{}^k[T]_{k+1} \right]
$$

We may define Homogeneous Axis-origin matrix for the purpose of reference.

$$
\begin{bmatrix} {}^i\{u\}_j & {}^i\{v\}_j & {}^i\{w\}_j & {}^i\{r\}_j \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^i[AO_H]_j
$$

Here, ${}^i[AO_H]_j$ is Homogeneous Axis-origin matrix of j-th frame as measured in i-th frame.

The idea may further be extended for computation based on base frame as:

$$
{}^1[AO_H]_j = {}^1[AO_H]_i\, {}^i[T]_j
$$

If i-th frame is the base frame numbered as frame 1, then $[AO_H]_1 = [I]$ and homogeneous position transformation matrix ${}^1[T]_j$ can be computed and can be interpreted as below.

$$
{}^1[AO_H]_j = {}^1[T]_j
$$

Thus,we may conclude:

1. If the link parameters and joint variables of a serial mechanism are known, homogeneous position transformation matrix may be used to compute homogenous axis-origin matrix for any frame from the homogeneous axis-origin matrix of other frame.

2. Homogenous position transformation matrix may be viewed as a matrix consisting of homogeneous unit vectors of the co-ordinate axes and homogeneous position vector of the origin of the frame with respect to base frame because homogenous axis-origin matrix of the base frame is an identity matrix.

Position analysis: Position of a general point fixed in j-th frame and given by position vector $\{R\}_j$ in j-th frame can be computed in 1st frame as follows to get ${}^1\{R\}_j$.

$$
\begin{aligned}
\{R\}_j &= \begin{bmatrix} R_x & R_y & R_z \end{bmatrix}^T \\
{}^1\{R\}_j &= R_x\, {}^1\{u\}_j + R_y\, {}^1\{v\}_j + R_z\, {}^1\{w\}_j + {}^1\{r\}_j \quad (1)
\end{aligned}
$$

So it can be related to homogenous position transformation matrix as follows.

$$\left\{\begin{array}{c} ^{1}\{R\}_j \\ 1 \end{array}\right\} = \left[\begin{array}{cccc} ^{1}\{u\}_j & ^{1}\{v\}_j & ^{1}\{w\}_j & ^{1}\{r\}_j \\ 0 & 0 & 0 & 1 \end{array}\right] \left\{\begin{array}{c} \{R\}_j \\ 1 \end{array}\right\} = {}^{1}[T]_j \left\{\begin{array}{c} \{R\}_j \\ 1 \end{array}\right\}$$

where, $\left\{\begin{array}{c} \{R\}_j \\ 1 \end{array}\right\}$ is called homogeneous position vector and is denoted by $\{R_H\}_j$.

2.1 Definition of joint motion property vector ${}^{1}\{D\}_i$:

1. For rotational joint motion,

$$^{1}\{D\}_i^R = \left\{\begin{array}{c} ^{1}\{e\}_i \\ ^{1}\{r\}_i \times {}^{1}\{e\}_i \end{array}\right\}$$

   for revolute joint j with respect to joint 1.

2. For translational joint motion,

$$^{1}\{D\}_i^T = \left\{\begin{array}{c} \{0\} \\ ^{1}\{e\}_i \end{array}\right\}$$

   for prismatic joint j with respect to joint 1.

The joint i and joint 1 have i-frame and base frame attached to them respectively. ${}^{1}\{e\}_i$ denotes unit vector along the motion axis of i-th frame as measured in base frame. ${}^{1}\{r\}_i$ denotes the relative position vector of the origin of i-th frame with respect to base frame as measured in base frame.

The concept may be explained further with reference to base frame (attached to base link) as below.

$$^{1}\{e\}_i = v4v3\left({}^{1}[T]_i \left\{\begin{array}{c} \{e\}_i \\ 0 \end{array}\right\}\right) = \left\{\begin{array}{c} e_x \\ e_y \\ e_z \end{array}\right\}$$

and

$$^{1}\{r\}_i = v4v3\left({}^{1}[T]_i \left\{\begin{array}{c} \{r\}_i \\ 1 \end{array}\right\}\right) = \left\{\begin{array}{c} r_x \\ r_y \\ r_z \end{array}\right\}$$

Quantities are measured in base frame.

The author has suggested the name for the vector because the vector contains all the information for joint motion and when multiplied with time-rate of joint variable it gives joint motion.

2.2 Matrix presentation of joint motion property vector:

1. for revolute joint i with respect to base frame.

$$\begin{aligned} ^{1}[D]_i^R &= v6m4\left({}^{1}\{D\}_i^R\right) \\ &= \left[\begin{array}{cc} ^{1}[\tilde{e}]_i & ^{1}\{r\}_i \times {}^{1}\{e\}_i \\ \lfloor 0 \rfloor & 0 \end{array}\right] \\ &= \left[\begin{array}{cccc} 0 & -e_z & e_y & r_y e_z - e_y r_z \\ e_z & 0 & -e_x & r_z e_x - e_z r_x \\ -e_y & e_x & 0 & r_x e_y - e_x r_y \\ 0 & 0 & 0 & 0 \end{array}\right] \end{aligned}$$

2. for prismatic joint i with respect to base frame.

$$\begin{aligned} ^{1}[D]_i^T &= v6m4\left({}^{1}\{D\}_i^T\right) = \left[\begin{array}{cc} [0] & ^{1}\{e\}_i \\ \lfloor 0 \rfloor & 0 \end{array}\right] \\ &= \left[\begin{array}{cccc} 0 & 0 & 0 & e_x \\ 0 & 0 & 0 & e_y \\ 0 & 0 & 0 & e_z \\ 0 & 0 & 0 & 0 \end{array}\right] \end{aligned}$$

So, it may be concluded that the unit vector $\{e\}_i$ representing the motion axis in i-th frame is to be measured in base frame to get ${}^{1}\{e\}_i$ as well as vector $\{r\}_i$ in i-th frame is also to be measured in base frame to get ${}^{1}\{r\}_i$. ${}^{1}\{e\}_i$ and ${}^{1}\{r\}_i$ can easily be computed using homogeneous position transformation matrix.

It requires mentioning that matrix similar to $[D]$-matrix is available in text books by Uicker [8]and Shahinpoor [7]. In those text books, the matrix was derived only from the concept of loop closure equation of serial manipulator robots and the expression derived by them was also different. They referred the matrix as $[D]$-matrix but they did not gave any specific name. With due respect to them, the author retained the symbol $[D]$ used by them but gave a name for the matrix. The author used his own approach for the development of the concept.

2.3 Some useful formulae from vector mechanics of serial robots:

All quantities are measured with respect to base frame i.e. frame 1.

Assuming that ${}^{1}\{e\}_i$ for i=1 to j-1 as motion axes and $\dot{\theta}_i$ as motion quantity for rotation , we get following formula for angular velocity of j-th link with respect to base frame.

$$^{1}\{\omega\}_j = {}^{1}\left\{\begin{array}{c} \omega_x \\ \omega_y \\ \omega_z \end{array}\right\}_j = \left[\sum_{i=1}^{j-1} \{\dot{\theta}_i \, {}^{1}\{e\}_i\}\right]$$

Using the concept of cross-product matrix for vector ${}^{1}\{\omega\}_j$ and ${}^{1}\{e\}_j$, we may write

$$^{1}[\tilde{\omega}]_j = \left[\sum_{i=1}^{j-1} [\dot{\theta}_i \, {}^{1}[\tilde{e}]_i]\right]$$

and

$$^{1}\{r\}_i = {}^{1}\{r\}_j - {}^{1}_{i}\{r\}_j$$

where ${}^{1}_{i}\{r\}_j$ indicates relative position vector of origin in j-th frame with respect the origin of the i-th frame as measured in base frame. ${}^{1}\{r\}_i$ and ${}^{1}\{r\}_j$ indicate the position vector of the origin of frame i and that of frame j as measured from base frame. The above relations are based on the vector mechanics of serial manipulator robots.

2.4 Development of concept of joint motion property vector / matrix with respect to base frame:

Differentiating the equation (1) and by assuming that ${}^{1}\{e\}_i$ for i=1 to j-1 as motion axes and $\dot{\theta}_i$ and $\dot{s}_i$ as corresponding motion quantity for rotation and translation, we get for a point fixed in j-th link,

$$\begin{aligned} {}^1\{\dot{R}\}_j &= R_x\,{}^1\{\omega\}_j \times {}^1\{u\}_j + R_y\,{}^1\{\omega\}_j \times {}^1\{v\}_j \\ &\quad + R_z\,{}^1\{\omega\}_j \times {}^1\{w\}_j + {}^1\{\dot{r}\}_j \end{aligned}$$

and

$$\begin{aligned} {}^1\{\dot{r}\}_j &= \sum_{i=1}^{j-1}\left[\dot{\theta}_i\,{}^1\{e\}_i \times_i^1 \{r\}_j + \dot{s}_i\,{}^1\{e\}_i\right] \\ &= \sum_{i=1}^{j-1}\left[\dot{\theta}_i\,{}^1\{e\}_i \times \left({}^1\{r\}_j - {}^1\{r\}_i\right) + \dot{s}_i\,{}^1\{e\}_i\right] \end{aligned}$$

we obtain

$$\begin{aligned} {}^1\{\dot{R}\}_j &= R_x\,{}^1\{\omega\}_j \times {}^1\{u\}_j + R_y\,{}^1\{\omega\}_j \times {}^1\{v\}_j \\ &\quad + R_z\,{}^1\{\omega\}_j \times {}^1\{w\}_j + {}^1\{\omega\}_j \times {}^1\{r\}_j \\ &\quad + \sum_{i=1}^{j-1}\left[{}^1\{r\}_i \times {}^1\{e\}_i\dot{\theta}_i\right] + \sum_{i=1}^{j-1}\left[\dot{s}_i\,{}^1\{e\}_i\right] \end{aligned}$$

$$\begin{aligned} {}^1\{\dot{R}\}_j &= {}^1\{\omega\}_j \times {}^1\{R\}_j + \sum_{i=1}^{j-1}\left[{}^1\{r\}_i \times {}^1\{e\}_i\dot{\theta}_i\right] \\ &\quad + \sum_{i=1}^{j-1}\left[\dot{s}_i\,{}^1\{e\}_i\right] \\ &= \left[\sum_{i=1}^{j-1}\left[\dot{\theta}_i\,{}^1\{e\}_i\right]\right] \times {}^1\{R\}_j \\ &\quad + \sum_{i=1}^{j-1}\left[{}^1\{r\}_i \times {}^1\{e\}_i\dot{\theta}_i\right] + \sum_{i=1}^{j-1}\left[\dot{s}_i\,{}^1\{e\}_i\right] \end{aligned}$$

$$\begin{aligned} \begin{Bmatrix} {}^1\{\dot{R}\}_j \\ 1 \end{Bmatrix} &= \left[\sum_{i=1}^{j-1}\left[\dot{\theta}_i\begin{bmatrix} {}^1[\tilde{e}]_i & {}^1\{r\}_i \times {}^1\{e\}_i \\ \lfloor 0 \rfloor & 0 \end{bmatrix}\right]\right]\begin{Bmatrix} {}^1\{R\}_j \\ 1 \end{Bmatrix} \\ &\quad + \left[\sum_{i=1}^{j-1}\left[\dot{s}_i\begin{bmatrix} [0] & {}^1\{e\}_i \\ \lfloor 0 \rfloor & 0 \end{bmatrix}\right]\right]\begin{Bmatrix} {}^1\{R\}_j \\ 1 \end{Bmatrix} \\ &= \left[\sum_{i=1}^{j-1}\left[{}^1[D]_i^R\,\dot{\theta}_i\right]\right]\begin{Bmatrix} {}^1\{R\}_j \\ 1 \end{Bmatrix} \\ &\quad + \left[\sum_{i=i}^{j-1}\left[{}^1[D]_i^T\,\dot{s}_i\right]\right]\begin{Bmatrix} {}^1\{R\}_j \\ 1 \end{Bmatrix} \end{aligned}$$

$${}^1\{\dot{R}_H\}_j = \left[\sum_{i=1}^{j-1}\left[{}^1[D]_i^R\,\dot{\theta}_i + {}^1[D]_i^T\,\dot{s}_i\right]\right]{}^1\{R_H\}_j$$

So for serial mechanisms having only single degree of freedom joint as $\dot{\phi}_i = \dot{\theta}_{i,i+1}\, or\;\; \dot{s}_{i,i+1}$, it may be generalized as

$${}^1\{\dot{R}_H\}_j = \left[\sum_{i=1}^{j-1}\left[{}^1[D]_i\dot{\phi}_i\right]\right]{}^1[T]_j\{R_H\}_j \qquad (2)$$

2.5 Definition of Derivatives of joint motion property vector and matrix form:

In vector form:
For rotational motion in revolute joint:

$${}^1\{\dot{D}\}_i^R = \begin{Bmatrix} {}^1\{\omega\}_i \times {}^1\{e\}_i \\ {}^1\{\dot{r}\}_i \times {}^1\{e\}_i + {}^1\{r\}_i \times \left({}^1\{\omega\}_i \times {}^1\{e\}_i\right) \end{Bmatrix}$$

For translational motion in prismatic joint:

$${}^1\{\dot{D}\}_i^T = \begin{Bmatrix} \{0\} \\ {}^1\{\omega\}_i \times {}^1\{e\}_i \end{Bmatrix}$$

In matrix form:

$${}^1[\dot{D}]_i^R = v6m4\left({}^1\{\dot{D}\}_i^R\right) \qquad {}^1[\dot{D}]_i^T = v6m4\left({}^1\{\dot{D}\}_i^T\right)$$

2.6 Homogeneous Velocity Transformation matrix:

From equation (2)

$$\frac{d}{dt}{}^1[T]_j\{R_H\}_j = \left[\sum_{i=1}^{j-1}\left[{}^1[D]_i\dot{\phi}_i\right]\right]{}^1[T]_j\{R_H\}_j$$

$$\left[\frac{d}{dt}{}^1[T]_j\right]{}^1[T]_j^{-1} = \sum_{i=1}^{j-1}\left[{}^1[D]_i\dot{\phi}_i\right] = {}^1[\Omega]_j$$

${}^1[\Omega]_j$ is called homogeneous velocity transformation matrix for j-th link with respect base frame.

$${}^1[\Omega]_j = \sum_{i=1}^{j-1}\left[{}^1[D]_i\dot{\phi}_i\right]$$

Here, $\dot{\phi}_i = \dot{\theta}_{i,i+1}\, or\, \dot{s}_{i,i+1}$

It may also be stated as follows

$${}^1\{\dot{R}_H\}_j = {}^1[\Omega]_j\,{}^1\{R_H\}_j = {}^1[\Omega]_j\,{}^1[T]_j\{R_H\}_j$$

2.7 Homogeneous Acceleration transformation matrix:

Differentiating above equation (2),

$${}^1\{\ddot{R}_H\}_j = \left[\sum_{i=1}^{j-1}{}^1[D]_i\ddot{\phi}_i + {}^1[\dot{D}]_i\dot{\phi}_i + {}^1[D]_i\,{}^1[\Omega]_j\dot{\phi}_i\right]{}^1[T]_j\{R_H\}_j$$

$$\begin{aligned} \left[\frac{d^2}{dt^2}{}^1[T]_j\right]{}^1[T]_j^{-1} &= \sum_{i=1}^{j-1}\left[{}^1[D]_i\ddot{\phi}_i + {}^1[\dot{D}]_i\dot{\phi}_i + {}^1[D]_i\,{}^1[\Omega]_j\dot{\phi}_i\right] \\ &= {}^1[A_H]_j \end{aligned}$$

${}^1[A_H]_j$ is called homogeneous acceleration transformation matrix for link j with respect to base frame. The derivatives of joint motion property vectors are discussed later in this paper.

The method may be extended to jerk computation also using the derivation formula. The formula for jerk computation is given below:

$${}^1\{\dddot{R}_H\}_j = \left[\sum_{i=1}^{j-1}\left[{}^1[D]_i\dddot{\phi}_i + 2\,{}^1[\dot{D}]_i\ddot{\phi}_i + \left({}^1[\ddot{D}]_i + [C]\right)\dot{\phi}_i\right]\right]{}^1[T]_j\{R_H\}_j$$

$$ {}^1\{\dddot{R}_H\}_j = {}^1[J_H]_j\,{}^1[T]_j\,\{R_H\}_j $$

$$ [C] = 2\,{}^1[\dot{\Omega}]_j\,[D]_i + {}^1[D]_i\,{}^1[\dot{\Omega}]_j + {}^1[D]_i\,{}^1[\Omega]_j\,{}^1[\Omega]_j $$

${}^1[J_H]_j$ is called homogeneous jerk transformation matrix for link j with respect to base frame.

2.8 Mathematical derivation of an important relation relating Derivatives of joint motion property matrix (outline):

For rotational motion in revolute joint:
in vector form:

$$ {}^1\{\dot{D}\}_i^R = \left\{ \begin{array}{c} {}^1\{\omega\}_i \times {}^1\{e\}_i \\ {}^1\{\dot{r}\}_i \times {}^1\{e\}_i + {}^1\{r\}_i \times \left({}^1\{\omega\}_i \times {}^1\{e\}_i\right) \end{array} \right\} $$

in matrix form:

$$ \begin{aligned} {}^1[\dot{D}]_i^R &= v6m4\left({}^1\{\dot{D}\}_i^R\right) \\ &= \begin{bmatrix} \left[{}^1\widetilde{\{\omega\}_i \times {}^1}\{e\}_i\right] & {}^1\{\dot{r}\}_i \times {}^1\{e\}_i + {}^1\{r\}_i \times \left({}^1\{\omega\}_i \times {}^1\{e\}_i\right) \\ [0] & 0 \end{bmatrix} \\ &= \begin{bmatrix} {}^1[\tilde{\omega}]_i & {}^1\{\dot{r}\}_i - {}^1[\tilde{\omega}]_i\,{}^1\{r\}_i \\ [0] & 0 \end{bmatrix} \begin{bmatrix} {}^1[\tilde{e}]_i & {}^1\{r\}_i \times {}^1\{e\}_i \\ [0] & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} {}^1[\tilde{e}]_i & {}^1\{r\}_i \times {}^1\{e\}_i \\ [0] & 0 \end{bmatrix} \begin{bmatrix} {}^1[\tilde{\omega}]_i & {}^1\{\dot{r}\}_i - {}^1[\tilde{\omega}]_i\,{}^1\{r\}_i \\ [0] & 0 \end{bmatrix} \\ &= {}^1[\Omega]_i\,{}^1[D]_i^R - {}^1[D]_i^R\,{}^1[\Omega]_1 \end{aligned} $$

For translational motion in prismatic joint:
in vector form:

$$ {}^1\{\dot{D}\}_i^T = \left\{ \begin{array}{c} \{0\} \\ {}^1\{\omega\}_i \times {}^1\{e\}_i \end{array} \right\} $$

in matrix form:

$$ \begin{aligned} {}^1[\dot{D}]_i^T &= v6m4\left({}^1\{\dot{D}\}_i^T\right) \\ &= \begin{bmatrix} [0] & {}^1[\tilde{\omega}]_i\,{}^1\{e\}_i \\ [0] & 0 \end{bmatrix} \\ &= \begin{bmatrix} {}^1[\tilde{\omega}]_i & {}^1\{\dot{r}\}_i - {}^1[\tilde{\omega}]_i\,{}^1\{r\}_i \\ [0] & 0 \end{bmatrix} \begin{bmatrix} [0] & {}^1\{e\}_i \\ [0] & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} [0] & {}^1\{e\}_i \\ [0] & 0 \end{bmatrix} \begin{bmatrix} {}^1[\tilde{\omega}]_i & {}^1\{\dot{r}\}_i - {}^1[\tilde{\omega}]_i\,{}^1\{r\}_i \\ [0] & 0 \end{bmatrix} \\ &= {}^1[\Omega]_i\,{}^1[D]_i^T - {}^1[D]_i^T\,{}^1[\Omega]_1 \end{aligned} $$

In general,

$$ {}^1[\dot{D}]_i = {}^1[\Omega]_i\,{}^1[D]_i - {}^1[D]_i\,{}^1[\Omega]_1 \qquad (3) $$

2.9 Some additional results for computation of velocity, acceleration , jerk and jacobians and derivatives

$$ \begin{aligned} {}^1[\dot{D}]_i &= {}^1[\Omega]_i\,{}^1[D]_i - {}^1[D]_i\,{}^1[\Omega]_i \\ {}^1[\dot{\Omega}]_i &= \sum_{k=1}^{i-1}\left[{}^1[D]_k\,\ddot{\phi}_k + \left({}^1[\Omega]_k\,{}^1[D]_k - {}^1[D]^1[\Omega]_k\right)\dot{\phi}_k\right] \\ {}^1[\ddot{D}]_i &= {}^1[\dot{\Omega}]_i\,{}^1[D]_i + {}^1[\Omega]_i\,{}^1[\dot{D}]_i - {}^1[\dot{D}]_i\,{}^1[\Omega]_i - {}^1[D]_i\,{}^1[\dot{\Omega}]_i \\ {}^1[\ddot{\Omega}]_i &= \sum_{k=1}^{i-1}\left[{}^1[D]_k\,\dddot{\phi}_k + 2\,{}^1[\dot{D}]_k\,\ddot{\phi}_k + [A]\,\dot{\phi}_k\right] \end{aligned} $$

$$ [A] = \left[{}^1[\dot{\Omega}]_k\,{}^1[D]_k + {}^1[\Omega]_k\,{}^1[\dot{D}]_k - {}^1[\dot{D}]_k\,{}^1[\Omega]_k - {}^1[D]_k\,{}^1[\dot{\Omega}]_k\right] $$

For homogeneous velocity transformation matrix computation and its time-derivative:

$$ {}^1\{\Omega\}_j = [D]\,\{\dot{\phi}\} \qquad {}^1\{\dot{\Omega}\}_j = [D]\,\{\ddot{\phi}\} + [\dot{D}]\,\{\dot{\phi}\} $$

for velocity computation:

$$ {}^1\left\{ \begin{array}{c} \omega \\ v_p \end{array} \right\}_j = \begin{bmatrix} d_1^a & \cdots & d_i^a & \cdots & d_{j-1}^a \\ d_1^l & \cdots & d_i^l & \cdots & d_{j-1}^l \end{bmatrix} \{\dot{\phi}\} = {}^1[\mathbf{J}]_j\,\{\dot{\phi}\} $$

For acceleration computation:

$$ \begin{aligned} {}^1\left\{ \begin{array}{c} \alpha \\ a_p \end{array} \right\}_j &= \begin{bmatrix} d_1^a & \cdots & d_i^a & \cdots & d_{j-1}^a \\ d_1^l & \cdots & d_i^l & \cdots & d_{j-1}^l \end{bmatrix} \{\ddot{\phi}\} \\ &\quad + \begin{bmatrix} dd_1^a & \cdots & dd_i^a & \cdots & dd_{j-1}^a \\ dd_1^l & \cdots & dd_i^l & \cdots & dd_{j-1}^l \end{bmatrix} \{\dot{\phi}\} \\ &= {}^1[\mathbf{J}]_j\,\{\ddot{\phi}\} + {}^1[\dot{\mathbf{J}}]_j\,\{\dot{\phi}\} \end{aligned} $$

For Jerk Computation:

$$ \begin{aligned} {}^1\left\{ \begin{array}{c} \dot{\alpha} \\ \dot{a}_p \end{array} \right\}_j &= \begin{bmatrix} d_1^a & \cdots & d_i^a & \cdots & d_{j-1}^a \\ d_1^l & \cdots & d_i^l & \cdots & d_{j-1}^l \end{bmatrix} \{\dddot{\phi}\} \\ &\quad + 2\begin{bmatrix} dd_1^a & \cdots & dd_i^a & \cdots & dd_{j-1}^a \\ dd_1^l & \cdots & dd_i^l & \cdots & dd_{j-1}^l \end{bmatrix} \{\ddot{\phi}\} \\ &\quad + \begin{bmatrix} ddd_1^a & \cdots & ddd_i^a & \cdots & ddd_{j-1}^a \\ ddd_1^l & \cdots & ddd_i^l & \cdots & ddd_{j-1}^l \end{bmatrix} \{\dot{\phi}\} \\ &= {}^1[\mathbf{J}]_j\,\{\dddot{\phi}\} + 2\,{}^1[\dot{\mathbf{J}}]_j\,\{\ddot{\phi}\} + {}^1[\ddot{\mathbf{J}}]_j\,\{\dot{\phi}\} \end{aligned} $$

Here ${}^1[\mathbf{J}]_j$ indicate the jacobian for j-th link with respect to base frame.

where

$$ \begin{aligned} [D] &= \left[{}^1\{D\}_1 \cdots {}^1\{D\}_i \cdots {}^1\{D\}_{j-1}\right] \\ [\dot{D}] &= \left[{}^1\{\dot{D}\}_1 \cdots {}^1\{\dot{D}\}_i \cdots {}^1\{\dot{D}\}_{j-1}\right] \\ \{\dot{\phi}\} &= \left[\begin{array}{ccccc} \dot{\phi}_1 & \cdots & \dot{\phi}_i & \cdots & \dot{\phi}_{j-1} \end{array}\right]^T \\ \{\ddot{\phi}\} &= \left[\begin{array}{ccccc} \ddot{\phi}_1 & \cdots & \ddot{\phi}_i & \cdots & \ddot{\phi}_{j-1} \end{array}\right]^T \\ \{\dddot{\phi}\} &= \left[\begin{array}{ccccc} \dddot{\phi}_1 & \cdots & \dddot{\phi}_i & \cdots & \dddot{\phi}_{j-1} \end{array}\right]^T \end{aligned} $$

$$ \begin{aligned} d_i^a &= m4v3\left({}^1[D]_i\right) \\ d_i^l &= v4v3\left({}^1[D]_i\,{}^1[T]_j\,\{R\}_j\right) \\ dd_i^a &= m4v3\left({}^1[\dot{D}]_i\right) \\ dd_i^l &= v4v3\left(\left[{}^1[\dot{D}]_i + {}^1[D]_i\,{}^1[\Omega]_j\right]\,{}^1[T]_j\,\{R_H\}_j\right) \\ ddd_i^a &= m4v3\left({}^1[\dot{\Omega}]_i\,{}^1[D]_i + {}^1[\Omega]_i\,{}^1[\dot{D}]_i - {}^1[\dot{D}]_i\,{}^1[\Omega]_i - {}^1[D]_i\,{}^1[\dot{\Omega}]_i\right) \\ ddd_i^l &= v4v3\left([[A]+[B]]\,{}^1[T]_j\,\{R_H\}_j\right) \\ [A] &= \left[{}^1[\dot{\Omega}]_i\,{}^1[D]_i + {}^1[\Omega]_i\,{}^1[\dot{D}]_i - {}^1[\dot{D}]_i\,{}^1[\Omega]_i - {}^1[D]_i\,{}^1[\dot{\Omega}]_i\right] \\ [B] &= \left[2\,{}^1[\dot{D}]_i\,{}^1[\Omega]_j + {}^1[D]_i\,{}^1[\dot{\Omega}]_j + {}^1[D]_i\,{}^1[\Omega]_j\,{}^1[\Omega]_j\right] \end{aligned} $$

3 Numerical example problems and Results

The DH-parameters for a 5-dof six-linked serial mechanism are presented below. Each joint is single-degree-of-freedom joint.

in mm	*in degree*	*in degree*	*in mm*
$a(1,2)=0$	$\alpha(1,2)=90$	$\theta(1,2)=30$	$s(1,2)=195$
$a(2,3)=178$	$\alpha(2,3)=0$	$\theta(2,3)=60$	$s(2,3)=0$
$a(3,4)=178$	$\alpha(3,4)=0$	$\theta(3,4)=-30$	$s(3,4)=0$
$a(4,5)=0$	$\alpha(4,5)=90$	$\theta(4,5)=0$	$s(4,5)=0$
$a(5,6)=0$	$\alpha(5,6)=0$	$\theta(5,6)=0$	$s(5,6)=97$

The schematic mechanism bears similarity in shape and size with the Microbot model TCM five-axis robot mentioned by Uicker [8] in text book but the joint motion characteristics have been modified to make it a suitable numerical example problem for describing all the features of analytical approach. The base link is numbered as link 1. The values of joint variables have been chosen arbitrarily.

Joint angle and joint distance are joint variables for revolute joint and prismatic joint respectively. In the table, $\theta(1,2)$,$\theta(2,3)$, $\theta(3,4)$ and $\theta(4,5)$ and $s(5,6)$ joint variables and their values have been chosen arbitrarily. The values $s(1,2)$, $s(2,3)$, $s(3,4)$, $s(4,5)$ and $\theta(5,6)$ are fixed quantities. The link 5 that connects 4-th coordinate frame and 5-th coordinate frame is of zero link length and zero joint distance.

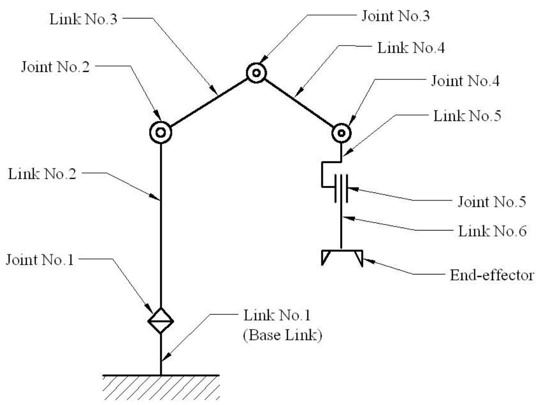

Figure 2: Kinematic Diagram of Schematic Mechanism

In figure 2: Joint No.1 - axial revolute joint; Joint no.2, joint no.3 and joint no.4 - normal revolute joints; Joint no.5 - prismatic joint;

Location point for analysis: point P.
Point P is a fixed point in link no.6 and is specified by the position vector (0,0,64mm) in 6-th frame that is attached to link no.6.

3.1 First Example Problem:

Joint types and Joint variable rates

$JointNo.[Type]$	$\dot{\phi}$	$\ddot{\phi}$	$\dddot{\phi}$
1 $[Revolute]$	$0.20 rad/s$	$0 rad/s^2$	$0 rad/s^3$
2 $[Revolute]$	$0 rad/s$	$0 rad/s^2$	$0 rad/s^3$
3 $[Revolute]$	$0 rad/s$	$0 rad/s^2$	$0 rad/s^3$
4 $[Revolute]$	$-0.35 rad/s$	$0 rad/s^2$	$0 rad/s^3$
5 $[Prismatic]$	$6 mm/s$	$0 mm/s^2$	$0 mm/s^3$

The values for $\dot{\phi}$ are arbitrarily chosen.

Results for a point P(0,0,64mm) in 6-th link as measured in base frame

position	ω	v_p	α	a_p	$\dot{\alpha}$	$\dot{a}_p$
mm	rad/s	mm/s	rad/s^2	mm/s^2	rad/s^3	mm/s^3
280.29	−0.175	−72.029	−0.606	−13.741	0.007	14.32
261.82	0.303	33.157	−0.035	−29.088	−0.012	−4.07
298.72	0.200	−33.371	0.000	14.980	0.000	5.36

Here,
ω - Angular velocity and v_p - Linear velocity of a point,
α - Angular acceleration and a_p - Linear acceleration,
$\dot{\alpha}$ - Angular Jerk and $\dot{a}_p$ - Linear jerk

Jacobian for link 6 :

$$[\mathbf{J}]_6 = \begin{bmatrix} 0.000 & 0.500 & 0.500 & 0.500 & 0.000 \\ 0.000 & -0.866 & -0.866 & -0.866 & 0.000 \\ 1.000 & -0.000 & -0.000 & -0.000 & 0.000 \\ -161.826 & -89.826 & 43.673 & 120.749 & 0.433 \\ 280.291 & -51.861 & 25.215 & 69.715 & 0.250 \\ 0.000 & 323.652 & 234.652 & 80.500 & -0.866 \end{bmatrix}$$

First Time-derivative of Jacobian for link 6 :

$$[\dot{\mathbf{J}}]_6 = \begin{bmatrix} 0.000 & 0.173 & 0.173 & 0.173 & 0.000 \\ 0.000 & 0.100 & 0.100 & 0.100 & 0.000 \\ 0.000 & 0.000 & -0.000 & 0.000 & 0.000 \\ -33.157 & 39.272 & 23.857 & 14.957 & -0.312 \\ -72.029 & -1.279 & 25.420 & 40.835 & -0.064 \\ 0.000 & -45.800 & -45.800 & -45.800 & -0.175 \end{bmatrix}$$

Second Time-derivative of Jacobian for link 6 :

$$[\ddot{\mathbf{J}}]_6 = \begin{bmatrix} 0.000 & -0.020 & -0.020 & -0.020 & 0.000 \\ 0.000 & 0.034 & 0.034 & 0.034 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 29.088 & -16.054 & -21.394 & -24.477 & -0.009 \\ -13.741 & 6.144 & 3.061 & 1.281 & -0.145 \\ 0.000 & -13.498 & -13.498 & -13.498 & 0.106 \end{bmatrix}$$

3.2 Second Example Problem:

Joint types and Joint variable rates:

$JointNo.[Type]$	$\dot{\phi}$	$\ddot{\phi}$	$\dddot{\phi}$
1 $[Revolute]$	$0.20 rad/s$	$0.20 rad/s^2$	$0.20 rad/s^3$
2 $[Revolute]$	$0 rad/s$	$0.20 rad/s^2$	$0.20 rad/s^3$
3 $[Revolute]$	$0 rad/s$	$0.20 rad/s^2$	$0.20 rad/s^3$
4 $[Revolute]$	$-0.35 rad/s$	$0.20 rad/s^2$	$0.20 rad/s^3$
5 $[Prismatic]$	$6.0 mm/s$	$5 mm/s^2$	$5 mm/s^3$

The values for $\dot{\phi}$, $\ddot{\phi}$ and $\dddot{\phi}$ are arbitrarily chosen.

Results for a point P(0,0,64mm) in 6-th link as measured in base frame

position	ω	v_p	α	a_p	$\dot{\alpha}$	$\dot{a}_p$
mm	rad/s	mm/s	rad/s^2	mm/s^2	rad/s^3	mm/s^3
280.29	−0.17	−72.03	0.24	−29.02	0.45	21.31
261.82	0.30	33.16	−0.55	36.83	−0.44	56.64
298.72	0.20	−33.37	0.20	138.41	0.20	43.72

Jacobian for link 6 :

$$[\mathbf{J}]_6 = \begin{bmatrix} 0.000 & 0.500 & 0.500 & 0.500 & 0.000 \\ 0.000 & -0.866 & -0.866 & -0.866 & 0.000 \\ 1.000 & -0.000 & -0.000 & -0.000 & 0.000 \\ -161.826 & -89.826 & 43.673 & 120.749 & 0.433 \\ 280.291 & -51.861 & 25.215 & 69.715 & 0.250 \\ 0.000 & 323.652 & 234.652 & 80.500 & -0.866 \end{bmatrix}$$

First Time-derivative of Jacobian for link 6 :

$$[\dot{\mathbf{J}}]_6 = \begin{bmatrix} 0.000 & 0.173 & 0.173 & 0.173 & 0.000 \\ 0.000 & 0.100 & 0.100 & 0.100 & 0.000 \\ 0.000 & 0.000 & -0.000 & 0.000 & 0.000 \\ -33.157 & 39.272 & 23.857 & 14.957 & -0.312 \\ -72.029 & -1.279 & 25.420 & 40.835 & -0.064 \\ 0.000 & -45.800 & -45.800 & -45.800 & -0.175 \end{bmatrix}$$

Second Time-derivative of Jacobian for link 6 :

$$[\ddot{\mathbf{J}}]_6 = \begin{bmatrix} 0.000 & 0.153 & 0.153 & 0.153 & 0.000 \\ 0.000 & 0.134 & 0.134 & 0.134 & 0.000 \\ 0.000 & 0.000 & -0.000 & -0.000 & 0.000 \\ -36.833 & -112.576 & -117.916 & -76.499 & 0.390 \\ -29.022 & -73.536 & -41.019 & 3.446 & 0.200 \\ 0.000 & 6.228 & 37.059 & 72.659 & 0.406 \end{bmatrix}$$

3.3 Discussion on results:

1. The results presented above are obtained from a computer program developed by the the author on the basis of the formulae discussed.

2. For first example problem, the results of the analysis for position, linear velocity, angular velocity linear acceleration and angular acceleration are verified with an other program also developed by author using the formulae available in text books by Saha [4] and Jazar [2].

3. For second example problem, no results are verified for want of of suitable formulae in standard text.

4. The author could not verify the results of jerk analysis because he could not find suitable formulae for jerk analysis in standard text.

5. The results for jacobian matrix are verified with corresponding values of angular velocities and linear velocities.

6. It appears from the results that the values for angular velocities and linear velocities are affected by first-order time-rate of variation of joint variables only but completely independent of the higher-order time-rate of variation of joint variables. These are expected from theory of kinematics of robotics.

7. Author has verified that the values of angular velocity, linear velocity, angular acceleration and linear acceleration are affected by first-order time-rate of variation and second time-rate of variation for joint variables but completely independent of higher-order time-derivative of joint variables. These are expected from theory of robotics.

3.4 Conclusions:

1. The method is well suited for kinematic analysis for the time-derivative of any order for joint variables. It is capable of handling various conditions of variation of joint variables. Complete generalized analysis is possible for serial robots and other serial mechanisms having any number of links.

2. The method can be effectively programmed.

3. The method may be used to lower pair of joints and can be modified to suit for higher pair of joint.

4 Future scope of work:

The method may be extended to analysis the kinematics of other multi-body systems consisting of rigid bodies like closed chain system, parallel system etc. Currently the author is working in this area.

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